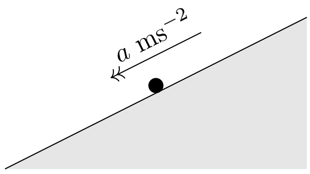
801. Find the range of $\mathrm{h}(x) = 64x^4 - 48x^2 + 5$, defined over the domain $\mathbb{R}$.

802. Show that $\left(\sqrt{2}+\sqrt{3}\right)^4 = 49 + 20\sqrt{6}$.

803. A particle is placed on a smooth slope and released. The slope has angle of inclination θ. Air resistance is assumed to be negligible.



Prove that $a = g\sin\theta$.

804. Three integers are chosen, without replacement, from $\{1, 2, 3, 4, 5\}$. Find the probability that

(a) the 1 is not chosen,

(b) all three odd numbers are chosen.

805. A function has instruction

$$\mathrm{f}(x) = x\sqrt{x} + 3x.$$

Determine whether, at $x = 1$, $\mathrm{f}(x)$ is increasing, decreasing or neither.

806. Consider the equation $\left(a + b\sqrt{6}\right)^2 = 58 - 12\sqrt{6}$, where a and b are integers.

(a) By multiplying out and equating coefficients, find simultaneous equations linking a and b.

(b) Hence, simplify $\sqrt{58 - 12\sqrt{6}}$.

807. Determine which of the two points $O : (0, 0)$ and $A : (4, 0)$ is closer to the line $y = 2x - 3$.

808. A student writes as follows:

"Friction acts to oppose the motion or possible motion of two surfaces in contact. When a car accelerates forwards, the motion that would occur were there no friction is the slipping of the tyres backwards across the surface of the road. Hence, friction acts forwards on the tyres, and drives the car forwards."

True or false?

809. A parabola is given by $x = y^2 - y + 20$.

(a) Determine all axis intercepts.

(b) By finding dx/dy and setting it to zero, find the coordinates of the point where the tangent is in the y direction.

(c) Hence, sketch the curve.

810. Solve the equation $\displaystyle\sum_{r=1}^{3} \frac{r - x}{rx} = \frac{1}{2}$.

811. An inequality is given as

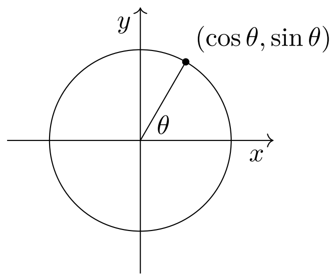
$$(x - a)(x - b)(x - c) > 0,$$

where a, b, c are constants such that $a < b < c$. By sketching a cubic graph, or otherwise, solve the inequality, giving your answer in set notation.

812. The graph $y = ax^4 + bx$, for constants a and b, is transformed onto the graph $x = ay^4 + by$. Describe this transformation.

813. Find the probability that three consecutive dice rolls give three different scores.

814. A unit circle is depicted, with the definitions of sin, cos and tan shown.



$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

From the definitions above, prove that

(a) $\sin^2\theta + \cos^2\theta \equiv 1$,

(b) $\tan^2\theta + 1 \equiv \sec^2\theta$.

815. Two vector lines have parametric equations

$$\mathbf{r}_1 = \begin{pmatrix} 1 + 4s \\ 5 - 2s \end{pmatrix}, \quad \mathbf{r}_2 = \begin{pmatrix} 6 - 2t \\ 13 + t \end{pmatrix}.$$

Determine whether or not the lines are parallel.

816. A sample of size 100 is taken, which has mean 21.2. Subsequently, a set of 10 data, whose mean is 19.4, is removed from the sample. Calculate the raised mean of the remaining 90 data.

817. Show that the greatest possible value of the quartic expression $20x^2(40 - x^2)$ is 8000.

818. You are given that invertible functions f and g have $\mathrm{f}(a) = b$, $\mathrm{f}(c) = d$, $\mathrm{g}(a) = c$ and $\mathrm{g}(b) = d$. Find the values of

(a) $\mathrm{gf}(a)$,

(b) $\mathrm{f}^{-1}\mathrm{g}^{-1}(d)$,

(c) $\mathrm{g}^{-1}\mathrm{fg}(a)$.

819. Prove that the exterior angles of a polygon add to 2π radians.

820. An arithmetic sequence has nth term

$$u_n = a + (n-1)d.$$

A new sequence w_n, for constants p, q, is given as

$$w_n = pu_n + qu_{n+1}.$$

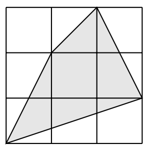
(a) Show that $qu_{n+1} = qu_n + qd$.

(b) Hence, show that w_n is also arithmetic, giving an expression for the common difference.

821. Two objects are dropped from rest simultaneously. Neglecting air resistance, prove that the distance between them remains constant until they land.

822. The two quadratic graphs $y = 3(x-2)(x-4)$ and $y = 3x^2 + 24x + 45$ are drawn on the same set of axes. Show that one is a reflection of the other, and find the equation of the line of symmetry.

823. The grid shown below consists of squares:



Find the fraction of the total area which is shaded.

824. It is not possible to use the factor theorem, over the real numbers, to establish whether $2x^2 + x + 1$ is a factor of $8x^5 + 12x^2 + 3$. Explain why.

825. Sketch the following graphs:

(a) $y = \sqrt{x^2}$,

(b) $y = (\sqrt{x})^2$.

826. Express the following as a polynomial in x:

$$(2x+3)^4 + (2x-3)^4.$$

827. The following definite integrals have value zero. In each case, shade the relevant regions whose areas cancel to give zero, marking them either + or − according to their contribution.

(a) $\displaystyle\int_{-1}^{1} x\,dx$, (b) $\displaystyle\int_{0}^{3} x^2 - 2x\,dx$.

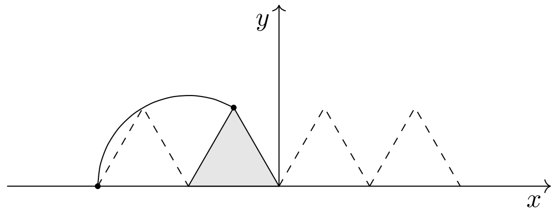
828. Solve $\sin x + \sqrt{3}\cos x = 0$ for $x \in [-180, 180°]$.

829. Separate the variables in the following differential equation, writing it in the form $\mathrm{f}(y)\frac{dy}{dx} = \mathrm{g}(x)$ for some functions f and g:

$$\frac{dy}{dx} - 4xy = 0.$$

830. You are given that the interior angles of a triangle are in AP. Determine the sum of the smallest and largest angles.

831. A *cyclogon* is produced by rolling an equilateral triangle of side length 1 along the x axis, marking the path of one of its vertices.



Show that each period has arc length $\frac{4\pi}{3}$.

832. Prove that no three distinct points on a parabola are collinear.

833. A parachutist of mass 55 kg is descending, under canopy, with a constant speed of 1.5 ms^{-1}. When she lands, she comes to rest in 0.4 seconds. Motion may be assumed to be purely vertical throughout.

(a) Draw a force diagram for the parachutist in the period before landing.

(b) Show that, at this stage, the parachute exerts an upwards force of 539 Newtons.

(c) Assuming the force exerted by the parachute remains constant throughout the motion, draw a force diagram during the landing.

(d) Determine the average force exerted on the parachutist's feet by the ground during the 0.4 seconds of the landing.

834. Shade the region of the (x, y) plane which satisfies both of the inequalities $x < 2$ and $y \geq 3$.

835. Write down the largest real domains over which the following functions may be defined:

(a) $x \mapsto \sqrt{x}$,

(b) $x \mapsto \sqrt{x^2}$,

(c) $x \mapsto \sqrt{x^3}$.

836. Two functions f and g have constant derivatives. Determine the possible numbers of roots of the equation $\mathrm{f}(x) = \mathrm{g}(x)$.

837. The parabola P_1, given by the equation $y = x^2$, is translated by the vector $k\mathbf{i}$ onto parabola P_2. The curves P_1 and P_2 intersect at $y = 16$. Find all possible values of k.

838. A student claims that $x(x^2+1)$ can be integrated factor by factor, as follows:

$$\int x(x^2+1)\,dx = \tfrac{1}{2}x^2\left(\tfrac{1}{3}x^3+x\right)+c.$$

Show that this claim is incorrect.

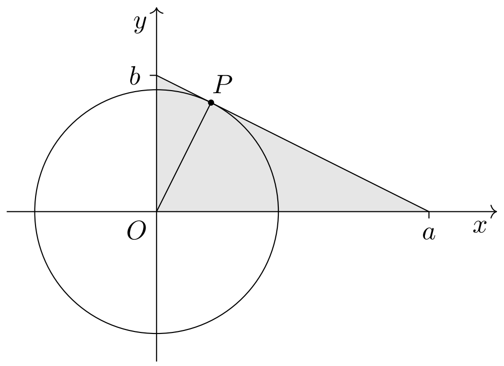
839. According to an ancient legend, an Indian vizier, who made a chessboard for the king, requested the following payment: 1 grain of rice on the first square, 2 on the second, 4 on the third, and so on. The king readily agreed to this. Taking the mass of a grain of rice to be a hundredth of a gram, show that, unbeknownst to the king, the vizier was, in fact, asking for 184 billion tonnes of rice.

840. Show that the following equation has no roots:

$$\frac{1}{1+\sqrt{x}}+\frac{1}{1-\sqrt{x}}=1.$$

841. Two distinct integers x_1 and x_2 are chosen from $\{x \in \mathbb{Z} : 1 \leq x \leq 5\}$. Find $\mathbb{P}(x_1x_2 < 10)$.

842. A right-angled triangle with side lengths (a, b, c) is placed with its perpendicular sides along x and y axes. A circle is drawn, centred at O, which is tangent to the hypotenuse at point P.



(a) Find, with coefficients in terms of a and b, the equation of
   i. the hypotenuse,
   ii. the radius.

(b) Hence, show that point P has coordinates

$$\left(\frac{ab^2}{c^2}, \frac{a^2b}{c^2}\right).$$

(c) Hence, prove that the circle has area

$$A_{\text{circle}} = \frac{\pi a^2b^2}{c^2}.$$

843. State, with a reason, whether the following gives a well-defined function:

$$\mathrm{g}: \begin{cases} \mathbb{R} \mapsto \mathbb{R} \\ x \mapsto \dfrac{1}{x^2+x+1}. \end{cases}$$

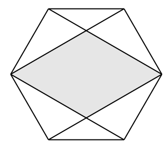
844. On the same axes, for constants $p, q > 0$ and $k < 0$, sketch the graphs

(a) $\dfrac{y-q}{x-p} = k$,

(b) $\dfrac{y+q}{x+p} = k$.

845. Show that $1+x^2$ is a factor of $1+x+x^2+x^3$.

846. The regular hexagon below has side length 1.



Show that the area of the shaded region is $\frac{2\sqrt{3}}{3}$.

847. You are given that $\mathbf{a}$ and $\mathbf{b}$ are two perpendicular unit vectors. Prove that $\frac{3}{5}\mathbf{a}+\frac{4}{5}\mathbf{b}$ and $-\frac{4}{5}\mathbf{a}+\frac{3}{5}\mathbf{b}$ are also perpendicular unit vectors.

848. Find all possible values of k satisfying

$$\int_1^k \frac{x^2+1}{x^2}\,dx = 4.8.$$

849. A projectile is launched from ground level at speed u, at an angle that will attain maximum range. Show that the greatest height in this motion is

$$h = \frac{u^2}{4g}.$$

850. The line $y = 2x+5$ is closest to $(0, 0)$ at point P. Find the coordinates of point P.

851. Two of the following statements are true; the other is not. Prove the two and disprove the other.

(a) $x^3(2^x+4) = 0 \implies x = 0$,

(b) $x^3(2^x-1) = 0 \implies x = 0$,

(c) $x^3(2^x-3) = 0 \implies x = 0$.

852. Solve $\sin^4 x + \sin^2 x = 0$ for $x \in [0, 360°)$.

853. A function $x \mapsto \mathrm{g}(x)$ has domain $[0, 1]$ and range $[0, 1]$. State, with a reason, whether the following are well-defined functions over the domain $[0, 1]$:

(a) $x \mapsto \mathrm{g}(2x)$,

(b) $x \mapsto 2\,\mathrm{g}(x)$.

854. Disprove the following statement: "Every pair of linear simultaneous equations in two unknowns x and y has a unique solution point (x, y)."

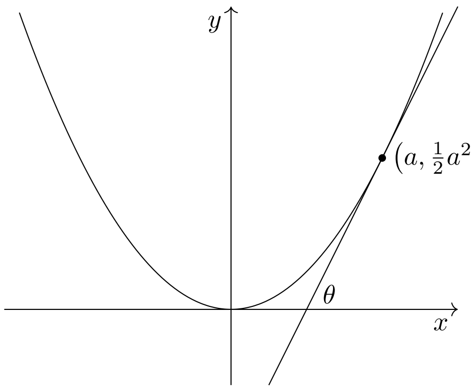
855. Without a calculator, evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \sin x \, dx$.

856. Show that the following equation defines a circle minus a point, and determine its centre and radius:

$$\frac{x+y}{x^2+y^2} = 1.$$

857. A die has been rolled. Determine whether the fact "The score is even" increases, decreases or doesn't affect the probability that the score is at least four.

858. The tangent to the curve $y = \frac{1}{2}x^2$ at $x = a$ makes an angle θ with the x axis:



Show that $\tan\theta = a$.

859. The graph $y = 12x(x-a)^2$, for $a > 0$, is a positive cubic with a double root at $x = a$.

(a) Sketch the graph.
(b) Show that the area enclosed by the graph and the x axis is a^4.

860. Find simplified expressions for the following sets, in which $a < b < c < d$ are constants:

(a) $(a, c) \cap [b, d]$,
(b) $(a, c] \cup [b, d)$,
(c) $(a, b] \cap [c, d)$.

861. The point $P : (3/2, \sqrt{3}/2)$ is rotated anticlockwise around the point $(1, 0)$ by $\frac{2\pi}{3}$ radians. Find the image of P under this transformation.

862. Solve the inequality $t^3 - 4t \geq 0$.

863. Variables x and y are related as follows:

$$y = \frac{1}{1+x} + \frac{1}{1-x}.$$

Make x the subject of this relationship.

864. Show that the following ratio cannot be expressed as $1 : \mathrm{p}(x)$, where $\mathrm{p}(x)$ is a polynomial:

$$x + 1 : x^3 - 4x^2 + 2x + 1.$$

865. A triangle has shortest side 4, and its larger two angles are $\arccos\frac{1}{8}$ and $\arccos\frac{9}{16}$.

(a) Find the smallest angle of the triangle.
(b) Hence, find its side lengths.

866. "The x and y coordinate axes are normal to the curve $x^2 + 4y^2 = 1$." True or false?

867. An object is in equilibrium, acted on by exactly three forces $\mathbf{F}, \mathbf{G}, \mathbf{H}$. Show that, if $\mathbf{F} = p\mathbf{G}$ for some $p \in \mathbb{R}$, then $\mathbf{H} = q\mathbf{G}$ for some $q \in \mathbb{R}$.

868. Solve $1.30x - 0.42 < \dfrac{0.88}{x}$.

869. A particular quadratic equation Q has n real roots. Its discriminant Δ satisfies $\Delta^2 - \Delta = 0$. Find all possible values of n.

870. In each case, state the value of the limit:

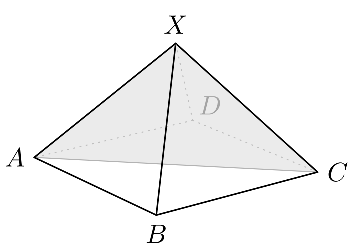
(a) $\displaystyle\lim_{x\to+\infty} \frac{x}{|x|+1}$,
(b) $\displaystyle\lim_{x\to-\infty} \frac{x}{|x|+1}$.

871. Show that the normal to $x = y^2$ at $(9, 3)$ crosses the x axis at $x = \frac{19}{2}$.

872. A projectile is launched from a height h above the ground, with fixed initial speed u.

(a) Prove that the landing speed will be the same whether the projectile is launched vertically upwards or downwards.
(b) Would the result in part (a) be different in the presence of air resistance?

873. The square-based pyramid shown below is formed of eight edges of unit length.



Find the area of the shaded triangle AXC.

874. Verify that the quartic curve $y = x^4$ satisfies the differential equation

$$\frac{d^2y}{dx^2} = 12\sqrt{y}.$$

875. A number generator in a computer produces the periodic sequence $1, 2, 3, 4, 5, 1, 2, \ldots$

   (a) Find the probability that 1 is chosen if a digit is randomly selected from among the first

      i. 5 terms,

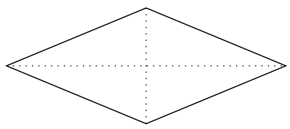
      ii. 6 terms,

      iii. $5k$ terms, $k \in \mathbb{N}$,

      iv. $5k + 1$ terms, $k \in \mathbb{N}$,

   (b) Show that iv. tends to $\frac{1}{5}$ as $k \to \infty$.

876. A rhombus is constructed, with interior angles of $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ radians, and side length l:



   (a) Use the cosine rule to show that the diagonals of the rhombus have lengths

$$d_1 = l\sqrt{2 - \sqrt{2}} \text{ and } d_2 = l\sqrt{2 + \sqrt{2}}.$$

   (b) Hence, show that the area of the rhombus is

$$A = \tfrac{\sqrt{2}}{2} l^2.$$

877. Show that $(x^2+4x+4)(x^2+x+3) = 0$ has exactly one real root.

878. Solve the following simultaneous equations:

$$x^2 + y^2 = 10,$$
$$2x^2 - 3y^2 = 15.$$

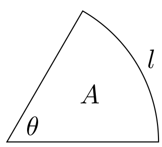
879. Functions f and g are said to *commute* if

$$\mathrm{f}(\mathrm{g}(x)) \equiv \mathrm{g}(\mathrm{f}(x)).$$

   (a) Show that, if the functions $\mathrm{f}(x) = x^2$ and $\mathrm{g}(x) = kx$ commute, then $k = 0$ or 1.

   (b) Prove that $\mathrm{f}(x) = x^a$ and $\mathrm{g}(x) = x^b$ always commute, for all $a, b \in \mathbb{R}$.

880. A sector has arc length l and subtends an angle θ, measured in radians, at the centre.



Show that the area is given by the formula

$$A = \frac{l^2}{2\theta}.$$

881. A student is attempting to solve the equation

$$(x - 2)^4 - (x - 2)^3 = 0.$$

His first step is to multiply out, using the binomial expansion. Explain why this is a long way round, and solve succinctly.

882. Given that $\cos 36° = \frac{1}{4}(1 + \sqrt{5})$, find a simplified expression for $\sec 36°$.

883. Determine the solution of $x^3 + 4x^2 - 27x - 90 = 0$.

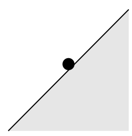
884. A region of the (x, y) plane is defined by

$$(x^2 + y^2 - 1)(x^2 + y^2 - 4) \le 0.$$

   (a) Explain, referring to the signs of the factors, why the region is an *annulus*, i.e. a thick ring bounded by two concentric circles.

   (b) Sketch and shade the region.

885. A particle is placed on a smooth slope of angle of inclination 45° and released from rest.



Neglecting air resistance, find the time taken for the particle to move 1 metre.

886. Simplify $\dfrac{\sqrt{2} + 1}{\sqrt{2} - 1} + \dfrac{\sqrt{2} - 1}{\sqrt{2} + 1}$.

887. The tangent to the curve $y = x^2 + 2x$ at point P passes through $(0, -4)$.

   (a) Write down the equation of a line, gradient m, which passes through the point $(0, -4)$.

   (b) Show that intersections of this line with the curve have x coordinates satisfying

$$x^2 + (2 - m)x + 4 = 0.$$

   (c) Using Δ, find all possible values of m.

   (d) Hence, find all possible coordinates of P.

888. A rational number can be expressed as $\frac{p}{q}$, where p and q are integers. Prove that the sum of two rational numbers is rational.

889. State which, if any, of the implications $\Longrightarrow$, $\Longleftarrow$ or $\Longleftrightarrow$ links the following statements concerning a real number x:

   ① $x^2 + x = 0$,

   ② $x + 1 = 0$.

890. An airliner is taking off. At the moment its wheels leave the runway, its instantaneous acceleration is $\left(\frac{1}{4}\mathbf{i} + \frac{1}{8}\mathbf{j}\right)g$ ms^{-2}, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in horizontal and vertical directions. Determine the magnitude, at this instant, of the contact force on a passenger of mass 60 kg.

891. Find the probability that, when two dice are rolled, at least one is a six.

892. A cubic function f is defined by

$$\mathrm{f}(x) = x^3 + x^2 - 16x - 20.$$

You are given that the equation $\mathrm{f}(x) = 0$ has a double root at $x = 2$.

(a) Solve $\mathrm{f}(x) = 0$.

(b) Find the gradient function $\mathrm{f}'(x)$.

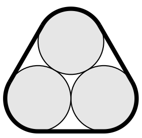
(c) Show that $\mathrm{f}'(x) = 0$ at the double root and $\mathrm{f}'(x) \neq 0$ at the single root.

(d) Interpret this result.

893. Exactly three forces act on an object, which is in equilibrium. The three forces have magnitudes $50, 60$ and F Newtons respectively. Determine the set of possible values of F.

894. Describe all functions f for which f' is linear.

895. Three cylindrical logs, each of radius r, are bound together with a loop of rope, as shown below in cross-section.



Find the total length of the loop of rope, giving an exact answer in terms of r.

896. Primes of the form $n! \pm 1$, for $n \in \mathbb{N}$, are known as *factorial primes*. A student claims that

"Every number of the form $n! \pm 1$ is prime."

Disprove the student's claim.

897. You are given that

$$\int_0^1 x^n (1 + x)(1 - x)\, dx = \tfrac{2}{35}.$$

Determine all possible values of n.

898. Prove the cosine rule.

899. Car A sets off from O with constant acceleration 0.4 ms^{-2}, along a straight road. Car B sets off two seconds later with constant acceleration 0.6 ms^{-2}, from a point 10 metres back from O. Both cars start from rest.

(a) Car B overtakes car A at time t. Show that t satisfies $t^2 - 12t - 88 = 0$.

(b) Find the distance that car B has travelled when it overtakes car A.

900. Three sets A, B, C are defined as $A = \{1, 2, ..., 10\}$, $B = \{1, 2, 3, 4, 5\}$ and $C = \{2, 4, 6, 8, 10\}$. Using a random number generator, an element X is picked from set A.

(a) Write down the value of

i. $\mathbb{P}(X \in B)$,

ii. $\mathbb{P}(X \in C)$,

iii. $\mathbb{P}(X \in B \cap C)$.

(b) By comparing these values, determine if the events represented by the sets B and C are independent.

——— End of 9th Hundred ———